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LETTER TO THE EDITOR

Real time functional effective action for the quantum dynamics of a transverse Ising model at finite temperature

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Abstract. The method of thermo-field dynamics (TFD, the real time, finite-temperature quantum field theory) for the quantum spin algebra has been employed to construct the coarse-grained functional action for the dynamics of the spin-$\frac{1}{2}$ Ising model in a transverse magnetic field. The applicability of the present approach to the unified treatment of the quantum and classical critical dynamics is pointed out.

There have recently been great efforts to explore the critical behaviour of quantum systems, motivated by the possibility of novel physical behaviour arising from the influence of quantum fluctuations on the phase transition picture (see Busiello *et al* (1983) for a review). A great deal of interest has been devoted to the study of the crossover phenomenon resulting from the interplay between classical and quantum behaviour in the low-temperature limit (De Cesare 1982, Lukierska-Walasek and Walasek 1983). The common feature of these approaches is the use of the imaginary time Matsubara (1955) technique to construct the generalised version of the Landau-Ginzburg-Wilson (LGW) functional for the subsequent renormalisation group (RG) analysis. However, as has been pointed out by Ruggiero and Zannetti (1983) the quantum-classical crossover in the critical dynamics cannot be described properly based on the 'imaginary time' technique alone. This difficulty stems from the fact that, in the Matsubara approach, the 'time' variable is restricted to a finite interval, so one requires discrete frequency summation and a tedious process of analytical continuation. Thus, the aim of the present letter is to approach this problem in terms of the real time functional framework, which allows us to study both quantum and classical critical dynamics in a unified way. To be specific we consider, as a particular example, the spin-$\frac{1}{2}$ Ising model in a transverse magnetic field which has been widely studied in the context of quantum critical phenomena (Elliot and Wood 1971, Young 1975, Pfeuty 1976, Lawrie 1978a, b, Lukierska-Walasek and Walasek 1983). The key point for our considerations is the use of the thermo-field dynamics (TFD) method due to Takahashi and Umezawa (1975), which has been successfully applied in various areas of physics (see Umezawa *et al* 1982) and has recently been generalised to the quantum spin algebra case (Whitehead *et al* 1984).

The Hamiltonian of the model has the form

$$H=-\tfrac{1}{2}\sum_{ij}J_{ij}S_i^zS_j^z-\Gamma\sum_i S_i^x \tag{1}$$

where S_i^α ($\alpha = x, y, z$) are the spin-$\frac{1}{2}$ operators obeying the usual commutation rules, while Γ denotes the transverse magnetic field. The dynamics of the system in TFD is determined by the thermal Hamiltonian $\hat{H}$ given by (Takahashi and Umezawa 1975)

$$\hat{H} = H - \tilde{H}$$

where $\tilde{H}$ is the tilde Hamiltonian, corresponding to the H Hamiltonian given by (1), written in terms of the tilde conjugate operators $\tilde{S}_i^\alpha$ associated with any S_i^α ($\tilde{H} \equiv H[\tilde{S}]$). The temperature enters the theory through the tilde substitution law (Umezawa *et al* 1982), being the basis of the Kubo–Martin–Schwinger relation, and controls the relation between S_i^α and $\tilde{S}_i^\alpha$ through their action on the temperature-dependent vacuum $|O(\beta)\rangle$

$$\langle O(\beta)|\boldsymbol{S}(t - \mathrm{i}\beta/2) = \langle O(\beta)|\tilde{\boldsymbol{S}}(t) \qquad \boldsymbol{S}(t + \mathrm{i}\beta/2)|O(\beta)\rangle = \tilde{\boldsymbol{S}}(t)|O(\beta)\rangle \tag{2}$$

where

$$\boldsymbol{S}(t) = \exp(\mathrm{i}\hat{H}t)\boldsymbol{S}\exp(-\mathrm{i}\hat{H}t).$$

In contrast to the derivation of the LGW functional via the 'imaginary time' technique, where one constructs the functional representation for the partition function, in the TFD method we start from the generating functional for the real time Green functions at finite temperature in the form

$$\begin{aligned} Z[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] &= \langle O(\beta)|\left(T \exp \mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t \hat{H}_\xi(t) \right)|O(\beta)\rangle \\ \hat{H}_\xi(t) &= \sum_i [\boldsymbol{\xi}_{1,i}(t)\boldsymbol{S}_{1,i}(t) + \boldsymbol{\xi}_{2,i}(t)\boldsymbol{S}_{2,i}(t)] \end{aligned} \tag{3}$$

where $\boldsymbol{\xi}_1(t)$ and $\boldsymbol{\xi}_2(t)$ denote the external sources associated with the non-tilde ($\boldsymbol{S}_1(t) \equiv \boldsymbol{S}(t)$) and tilde ($\boldsymbol{S}_2(t) \equiv \tilde{\boldsymbol{S}}(t)$) operators, taken in the Heisenberg picture

$$\boldsymbol{S}_a(t) = \exp(\mathrm{i}\hat{H}t)\boldsymbol{S}_a \exp(-\mathrm{i}\hat{H}t) \qquad (a = 1, 2) \tag{4}$$

and T is the usual time ordering operator.

In order to have a suitable form of the generating functional for the RG analysis we write the expression (3) as a functional integral by making use of the functional Fourier transform (Rzewuski 1969). Thus, one obtains

$$Z[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] = \int \mathrm{D}\boldsymbol{\varphi}_1 \mathrm{D}\boldsymbol{\varphi}_2 \exp\left(\mathrm{i}\mathscr{L}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2] + \mathrm{i} \int \mathrm{d}x \boldsymbol{\xi}_a(x)\boldsymbol{\varphi}_a(x) \right) \tag{5}$$

where

$$\int \mathrm{d}x \ldots = \sum_i \int_{-\infty}^{+\infty} \mathrm{d}t \ldots.$$

From (3) and (5) it becomes apparent that the physical quantities, which are expressed as the vacuum averages of products of the original spin operators, can be expressed via (5) in terms of the expectation values of products of the fields $\boldsymbol{\varphi}_a(t)$. The expectation value here means the functional averaging

$$\langle \ldots \rangle = \int \mathrm{D}\boldsymbol{\varphi}_1 \mathrm{D}\boldsymbol{\varphi}_2 \exp(\mathrm{i}\mathscr{L}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]) \ldots \tag{6}$$

with respect to the functional $\mathscr{L}[\varphi_1, \varphi_2]$ acting as a weighting factor. The latter is related to the $Z[\xi_1, \xi_2]$ via the inverse Fourier transform

$$\exp\{\mathrm{i}\mathscr{L}[\varphi_1, \varphi_2]\} = \int \mathrm{D}\xi_1 \mathrm{D}\xi_2 \exp\left(\mathrm{i}\, W[\xi_1, \xi_2] - \mathrm{i} \int \mathrm{d}x\, \xi_a(x)\varphi_a(x)\right) \tag{7}$$

where we have introduced the generating functional for the cumulant averages

$$\mathrm{i}\, W[\xi_1, \xi_2] = \ln Z[\xi_1, \xi_2]. \tag{8}$$

In principle the explicit form of the 'action functional' $\mathscr{L}[\varphi_1, \varphi_2]$ can be derived from the generating functional for cumulants by performing the functional integral (7). In practice this can be achieved by using the one-loop approximation, which is equivalent to the Gaussian average. However, by doing this, one immediately realises that the exponential factor on the left-hand side of (7) represents just the generating functional for the one-particle irreducible vertex functions. Thus, up to a constant factor, one obtains the effective action in a form of the Volterra series

$$\mathscr{L}[\varphi_1, \varphi_2] = \int \sum_{n=2}^{\infty} \frac{1}{n!} \mathrm{d}x_1 \ldots \mathrm{d}x_n \Gamma^{\alpha_1 \cdots \alpha_n}_{a_1 \ldots a_n}(x_1 \ldots x_n)\varphi^{\alpha_1}_{a_1}(x_1) \ldots \varphi^{\alpha_n}_{a_n}(x_n) \tag{9}$$

with the multipoint real time vertex functions $\boldsymbol{\Gamma}(x_1, \ldots, x_n)$ as the coefficients of the expansion. From the general structure of the TFD (Niemi and Semenoff 1983) it follows that the action (9) can always be separated into temperature-independent and temperature-dependent parts:

$$\mathscr{L}[\varphi_1, \varphi_2] = \mathscr{L}_0[\varphi_1] - \mathscr{L}_0[\varphi_2] + \mathscr{L}_\beta[\varphi_1, \varphi_2] \tag{10}$$

where all the temperature is contained in the last term of (10).

To analyse the critical phenomena it is sufficient to take into account the simplified version of the functional $\mathscr{L}[\varphi_1, \varphi_2]$ retaining in the expansion the terms up to fourth order only. Furthermore, the components of the field φ which become critical are $\overset{z}{\varphi} \equiv \varphi$ ones and the remaining may be integrated out. Then, for the two-point transverse vertex function in the expansion (9), one has explicitly

$$[\boldsymbol{\Gamma}^{zz}]^{-1}(\boldsymbol{k}, \omega) = \boldsymbol{K}^{zz}(\boldsymbol{k}, \omega) = \frac{\boldsymbol{\Sigma}^{zz}(\boldsymbol{k}, \omega)}{1 - \boldsymbol{\tau} J(\boldsymbol{k})\boldsymbol{\Sigma}^{zz}(\boldsymbol{k}, \omega)} \tag{11}$$

where the correlation function $\boldsymbol{K}^{zz}(\boldsymbol{k}, \omega)$ and self-energy part $\boldsymbol{\Sigma}^{zz}(\boldsymbol{k}, \omega)$ acquire the matrix form

$$\boldsymbol{K}^{zz} = \begin{pmatrix} K^{zz}_{11} & K^{zz}_{12} \\ K^{zz}_{21} & K^{zz}_{22} \end{pmatrix} \qquad \boldsymbol{\Sigma}^{zz} = \begin{pmatrix} \Sigma^{zz}_{11} & \Sigma^{zz}_{12} \\ \Sigma^{zz}_{21} & \Sigma^{zz}_{22} \end{pmatrix} \tag{12}$$

and

$$\boldsymbol{\tau} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the lowest-order approximation, using the Wick theorem for spins in the TFD formalism (Whitehead *et al* 1984) one gets for the self-energy part

$$\boldsymbol{\Sigma}^{zz}(\boldsymbol{k}, \omega) \simeq \boldsymbol{\Sigma}^{zz}_0(\omega) = -\tfrac{1}{2} B_{1/2}(\beta\Gamma/2)\boldsymbol{U}_B(\omega) \frac{\boldsymbol{\tau}\Gamma}{(\omega + \mathrm{i}\delta\boldsymbol{\tau})^2 - \Gamma^2} \boldsymbol{U}^+_B(\omega) \tag{13}$$

where $B_S(x)$ is the Brillouin function and δ is the positive infinitesimal, while $U_B(\omega)$ is the thermal transformation matrix for the boson field (Takahashi and Umezawa 1975)

$$U_B(\omega) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \qquad \cosh^2\theta = [1-\exp(-\beta\omega)]^{-1}. \tag{14}$$

In the long wavelength limit we can expand the Fourier transform of the exchange integral according to

$$J(\boldsymbol{k}) = J(0) - \tilde{J}k^2 + O(k^4). \tag{15}$$

Thus, in the continuum limit, we get for the effective action (9) the following simplified form:

$$\mathscr{L}[\varphi_1, \varphi_2] = \int dx \int dx' \, \varphi_a(x)[\boldsymbol{\Delta}(x-x')]^{-1}_{ab}\varphi_b(x') + u \int dx \, [\varphi_1^4(x) - \varphi_2^4(x)] \tag{16}$$

where, in (16), the leading temperature dependence enters only via the propagator $\boldsymbol{\Delta}(x-x')$, while the higher-order terms and the non-local character of the four-point spin vertex has been neglected. In the momentum-frequency representation the propagator $\boldsymbol{\Delta}(\boldsymbol{k}, \omega)$ becomes

$$\boldsymbol{\Delta}(\boldsymbol{k}, \omega) = \boldsymbol{U}_B(\omega)\boldsymbol{\tau}[m_0^2 + k^2 - c^2(\omega + i\delta\boldsymbol{\tau})^2]^{-1}\boldsymbol{U}_B^+(\omega) \tag{17}$$

where $c^{-2} = \frac{1}{2}\tilde{J}\Gamma B_{1/2}(\beta\Gamma/2)$ and m_0 is the 'critical mass' which vanishes along the critical line given by

$$\Gamma/J(0) = \tfrac{1}{2}\tanh(\beta\Gamma/2). \tag{18}$$

The functional (13) constitutes the particular example of the quantum counterpart of the classical Lagrangian formulation for the critical dynamics due to Jansen (1976) based on the Martin *et al* (1973) approach. In the latter, in order to obtain the proper description of the dynamics, the introduction of the second conjugate field(the so-called response field) is needed. In the present approach, which relies on the TFD, a similar role is played by the tilde conjugate field. As has been shown by Matsumoto *et al* (1984) the response theory can be put in a simple time-ordered TFD formulation. The reinterpretation of these results within the functional framework is then obvious. Because the components of the thermo-doublet field are mixed via the matrix elements of the propagator (17) only, one realises from (14) and (17) that, at $T = 0$, the TFD reduces simply to the duplication of the standard quantum field theory. Thus, after analytical continuation to the Matsubara 'imaginary time', the standard Euclidean version of the zero-temperature action for the transverse Ising model is retrieved. In conclusion we note that the formulation presented here can be regarded as the starting point for the unified analysis of the quantum and classical critical dynamics within the renormalisation group method. An explicit presentation of this subject will be given elsewhere.

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